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On Difference Schemes of Third Order Accuracy for Nonlinear Hyperbolic Systems

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In papers [1], [2] a scheme of third order accuracy was proposed for a quasilinear hyperbolic system of partial differential equations. The scheme had been constructed on the basis of the generalization of the Runge-Kutta method. The present paper gives further development and refinement of these results and extension to equations with three independent variables.

I. CONSTRUCTION OF THE THIRD ORDER DIFFERENCE SCHEME

Let us consider a nonlinear system of partial equations of the first order, written in divergent form

$$\frac{\partial w}{\partial t} = \frac{\partial F(w, x, t)}{\partial x} + f(w, x, t). \tag{1}$$

We shall assume that for the values of $w(x, t)$, x and t considered, that the system is hyperbolic; that is, the matrix $F_w = \partial F/\partial w$ has real and different eigenvalues.

Let us introduce in the x, t plane a mesh of size $\Delta x = h$, $\Delta t = \tau$ and denote

$$x_m = mh, \quad t^n = n\tau, \quad q = \tau/h, \quad w(x_m, t^n) = w_m{}^n, \quad w = \{w_m{}^n\}.$$

The numerical solution of the Cauchy problem for Eq. (1) reduces to an algorithm for the computation of w^{n+1}, assuming that w^n is known. We determine the w^{n+1} by means of an iterative process

$$\{w_m{}^n\} = w^n = w^{[0]} \rightarrow w^{[1]} \rightarrow \cdots \rightarrow w^{[R]} = w^{n+1} = \{w_m^{n+1}\}.$$

Let us construct the difference scheme in such a way that the value of w_m^{n+1} depends on five values of $w_m{}^n$, namely at the points with indexes m, $m \pm 1$, $m \pm 2$. Actually four points are sufficient for the third order scheme, but the five point scheme is more symmetric and suitable for general system with arbitrary characteristic directions.

Let us define the difference operators μ, δ, and the identity operator I as follows:

$$\mu\varphi_\epsilon = 2^{-1}(\varphi_{\epsilon+1/2} + \varphi_{\epsilon-1/2})$$
$$\delta\varphi_\epsilon = \varphi_{\epsilon+1/2} - \varphi_{\epsilon-1/2}; \qquad I\varphi_\epsilon = \varphi_\epsilon,$$

where ϵ is an integer or half-integer.

Let us put $R = 3$ and write the formulae for $w^{[1]}$, $w^{[2]}$, $w^{[3]}$ imposing only the conditions of symmetry (the notation is changed a little from [1], [2]):

$$\begin{aligned}
w_m^{[0]} &= w_m{}^n \\
w_{m+1/2}^{[1]} &= \mu w_{m+1/2}^{[0]} + \beta_{11}\{q\,\delta F + \tau\mu f\}_{m+1/2}^{[0]} \\
w_m^{[2]} &= (I + \omega_{20}\delta^2)\, w_m^{[0]} \\
&\quad + \beta_{21}\{q\mu\,\delta F + \tau(I + \theta_{21}\delta^2) f\}_m^{[0]} + \beta_{22}\{q\,\delta F + \tau\mu f\}_m^{[1]} \\
w_m^{n+1} = w_m^{[3]} &= (I + \omega_{30}\delta^2 + \gamma_{30}\delta^4)\, w_m^{[0]} \\
&\quad + \beta_{31}\{q(I + \omega_{31}\delta^2)\,\mu\,\delta F + \tau(I + \theta_{31}\delta^2 + \gamma_{31}\delta^4) f\}_m^{[0]} \\
&\quad + \beta_{32}\{q(I + \omega_{32}\delta^2)\,\delta F + \tau(I + \theta_{32}\delta^2)\,\mu f\}_m^{[1]} \\
&\quad + \beta_{33}\{q\mu\,\delta F + \tau(I + \theta_{33}\delta^2) f\}_m^{[2]}.
\end{aligned} \tag{2}$$

Here

$$\{q\mu^k\,\delta^l F\}_\epsilon^{[2]} = q\mu^k\,\delta^l F(w_\epsilon^{[2]}, x_\epsilon, t^n + \alpha_r\tau)$$

and an analogous expression holds for $\{\tau\mu^k\delta^l f\}_\epsilon^{[2]}$.

α_r, β_{rs}, ω_{rs}, θ_{rs}, γ_{rs} are parameters which must be selected so that the expansion of $w_m^{n+1} = w^{[3]}$ in powers of τ will be identical with the expansion

$$w_m^{n+1} = w_m{}^n + \tau(w_t)_m^n + \frac{\tau}{2!}(w_{tt})_m^n + \frac{\tau^3}{3!}(w_{ttt})_m^n + \cdots \tag{3}$$

to terms of order τ^3, inclusive.

It is assumed that τ and h are connected by the condition $q = \text{const}$.

The derivatives w_t, w_{tt}, and w_{ttt} are found by repeated differentiation of Eq. (1) with respect to t, and are expressed as follows:

$$\begin{aligned}
w_t &= D_x F + f = H \\
w_{tt} &= D_x\{F_w H + F_t\} + f_w H + f_t = K \\
w_{ttt} &= D_x\{F_w K + F_{ww} H^2 + 2F_{wt} H + F_{tt}\} \\
&\quad + f_w K + f_{ww} H^2 + 2f_{wt} H + f_{tt} = L
\end{aligned} \tag{4}$$

where

$$D_x\varphi(w, x, t) = \frac{\partial\varphi}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial\varphi}{\partial x} = \varphi_w w_x + \varphi_x.$$

By successively expanding $w^{[1]}_{m+1/2}$, $w^{[2]}_m$, $w^{[3]}_m$ in powers of τ, we obtain

$$
\begin{aligned}
w^{[1]}_{m+1/2} &= w + \beta_1\tau\left(H + \frac{1}{2q\beta_1} w_x\right) + O(\tau^2)\\
w^{[2]}_m &= w + \beta_2\tau H + \frac{\tau^2}{2!}\left\{2\beta_1\beta_{22}K + \frac{1}{q^2}\,\omega_{20}w_{xx}\right\} + O(\tau^3)\\
w^{[3]}_m &= w + \beta_3\tau H + \frac{\tau^2}{2}\left\{2(\beta_1\beta_{32} + \beta_2\beta_{33})\,K + \frac{2\omega_{30}}{q^2}\,w_{xx}\right\}\\
&\quad + \frac{\tau^3}{3!}\Big\{3(\beta_1{}^2\beta_{32} + \beta_2{}^2\beta_{33})\,L\\
&\quad + 3[2\beta_1\beta_{22}\beta_{33} - \beta_1{}^2\beta_{32} - \beta_2{}^2\beta_{33}][D_x(F_wK) + f_wK]\\
&\quad + \left[\beta_3 + 6\beta_{31}\omega_{31} + 6\beta_{32}\omega_{32} - \frac{3}{4}\beta_{32}\right]\frac{1}{q^2}\,D_x{}^3F\\
&\quad + \left[\frac{3}{4}\beta_{32} + 6\beta_{33}\omega_{20}\right]\frac{1}{q^2}\,[D_x(F_ww_{xx}) + f_ww_{xx}]\\
&\quad + \left[6(\beta_{31}\theta_{31} + \beta_{32}\theta_{32} + \beta_{33}\theta_{33}) + \frac{3}{4}\beta_{32}\right]\frac{1}{q^2}\,D_x{}^3f\Big\} + O(\tau^4),
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
\beta_1 &= \beta_{11} = \alpha_1\\
\beta_2 &= \beta_{21} + \beta_{22} = \alpha_2\\
\beta_3 &= \beta_{31} + \beta_{32} + \beta_{33} = \alpha_3\,.
\end{aligned}
$$

The determination of the expansion coefficients in Eq. (5) requires tiresome and unwieldy algebraic calculation. For this reason the reduction of formulae (5) was carried out automatically on the computer. This part of the work was done by Miss E. J. Nazhestkina. The program for the algebraic transformation was written in special algorithmic language REFAL ("the recursive function algebraic language") developed by V. F. Turchin [3]. The computer used was the BESM–6 with the REFAL–interpreter.

After comparing of expansions (5) and (3) and taking (4) into account we find that the value of $w^{(2)}_m$ approximates $w(x_m\,, t^n + d_2\tau)$ to an accuracy of order τ^2 if $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$ and

$$
\begin{aligned}
2\beta_1\beta_{22} &= \beta_2{}^2;\\
\omega_{20} &= 0.
\end{aligned}
\tag{6}
$$

The value of $w_m^{[3]}$ approximates $w_m^{n+1} = w(x_m, t^n + \tau)$ to an accuracy of τ^3 if

$$\begin{aligned} \beta_{31} + \beta_{32} + \beta_{33} &= 1 \\ 2(\beta_1\beta_{32} + \beta_2\beta_{33}) &= 1 \\ 6\beta_1\beta_{22}\beta_{33} &= 1 \\ 6(\beta_{31}\omega_{31} + \beta_{32}\omega_{32} + \beta_{33}\omega_{20}) &= -1; \\ \omega_{30} &= 0 \\ 3\beta_1{}^2\beta_{32} + 3\beta_2{}^2\beta_{33} &= 1 \\ \beta_{32} + 8\beta_{33}\omega_{20} &= 0 \\ 8(\beta_{31}\theta_{31} + \beta_{32}\theta_{32} + \beta_{33}\theta_{33}) + \beta_{32} &= 0. \end{aligned} \tag{7}$$

In order to obtain the third order scheme it is sufficient to satisfy the first five equations only if $F = AW$, where A is a constant matrix. It is necessary to satisfy the next two equations if A is not constant, or if F is a non-linear function of W. It is necessary to satisfy the last equation if Eqs. (1) are nonhomogeneous ($f \not\equiv 0$).

The coefficients γ_{30} and γ_{31} do not influence the accuracy of the results, but the value of γ_{30} must be chosen correctly for stability.

If we require that $w_m^{[2]}$ should have the second order accuracy, then from Eqs. (6) and (7) we have:

$$\begin{gathered} \omega_{20} = \beta_{32} = 0 \\ \beta_2 = 2/3; \qquad \beta_{31} = 1/4; \qquad \beta_{33} = 3/4; \qquad \omega_{31} = -2/3 \\ \theta_{31} + 3\theta_{33} = 0 \\ \beta_1\beta_{22} = 2/9; \qquad \beta_{21} + \beta_{22} = 2/3. \end{gathered} \tag{8}$$

If we set $\theta_{31} = \beta_{21} = 0$, we obtain the scheme given in [1] where an incorrect statement is made, namely that this scheme is unique. Actually the second order requirement for $w_m^{[2]}$ form a two-parametric family of third-order schemes.

II. Stability

To investigate stability let us consider scheme (2) for equation $w_t = aw_x$, where a is a constant. After substituting $w_{m+1/2}^{[1]}$ and $w_m^{[2]}$ in $w_m^{[3]}$, we obtain

$$w_m^{n+1} = \left\{ I + \sigma\mu\delta + \frac{\sigma^2}{2}\mu^2\delta^2 + \frac{\sigma^3 - \sigma}{6}\mu\delta^3 - \frac{\omega}{24}\delta^4 \right\} w_m{}^n, \tag{9}$$

where

$$\omega = -24\{\gamma_{30} + \beta_{11}\beta_{32}(\omega_{32} - \tfrac{1}{4})\}$$

$$\sigma = qa.$$

By using the Fourier method, we obtain the expression for the amplification factor $\lambda(\varphi)$, $\varphi = kh$:

$$\lambda(\varphi) = 1 - \frac{\sigma^2}{2}\sin^2\varphi - \frac{2\omega}{3}\sin^4\frac{\varphi}{2} + i\left\{\sigma\sin\varphi + \frac{2}{3}(\sigma - \sigma^3)\sin\varphi\sin^2\frac{\varphi}{2}\right\} \tag{10}$$

and

$$1 - |\lambda(\varphi)|^2 = \frac{4}{9}\{[4\sigma^2(1-\sigma^2)^2 - (\omega - 3\sigma^2)^2]z^2 - 2[2\sigma^2(1-\sigma^2)^2 + 3\sigma^2(\omega - \sigma^2 - 2)]z + 3(\omega - 4\sigma^2 + \sigma^4)\}z^2 = z^2P(z), \tag{11}$$

where $z = \sin^2\varphi/2$, $0 \leqslant z \leqslant 1$.

It follows from Eq. (11) that scheme (2) is stable for $w_t = aw_x$ if, and only if, for a given σ and ω the polynomial $P(z)$ is nonnegative for $0 \leqslant z \leqslant 1$. It easy to verify that this will be the case if, and only if,

$$|\sigma| \leqslant 1$$
$$4\sigma^2 - \sigma^4 \leqslant \omega \leqslant 3. \tag{12}$$

For the system for which $F = Aw$, where $A = \text{const}$ the conditions (12) remain valid if $\sigma = q\,|\,\xi\,|_{\max}$, where $|\,\xi\,|_{\max}$ is the maximum modulus of the eigenvalues of the matrix A.

A numerical check shows that for the general system (1), the third-order scheme (2) is stable if for all values of w, x, and t, the conditions (12) are valid with $\sigma = q\,|\,\xi\,|_{\max}$, where ξ is the eigenvalue of F_w.

III. Some Examples of Computations

In paper [2] some examples of calculations in gas dynamics using the third-order scheme were presented. They show that the application of scheme (2) to the numerical computation of discontinuous solutions is sufficiently effective. Comparison with the Lax–Wendroff scheme allows one to conclude that the third-order scheme is qualitatively the same near the discontinuity.

One of the advantages of scheme (2) is that it may be applied to any hyperbolic system of the form (1). In particular, one can compute without any trouble steady supersonic flow, which is described by the hyperbolic system:

$$\partial w/\partial x + \partial F/\partial y = 0,$$

where w and F are determined as follows (the coordinate x plays the role of *time*):

$$w = \begin{Bmatrix} \rho u \\ p + \rho u^2 \\ \rho uv \\ (e + p)\, u \end{Bmatrix}; \qquad F = -\begin{Bmatrix} \rho v \\ \rho uv \\ p + \rho v^2 \\ (e + p)\, v \end{Bmatrix},$$

where

$$e = \frac{\rho(u^2 + v^2)}{2} + \frac{1}{k - 1} \bar{P}$$

$$k = C_P/C_V = \text{const},$$

and where u, v are velocity components, $\bar{P}$ is pressure and ρ is density. The components of F are expressed in terms of the components of w, and it is easy to write explicit formulae.

Two examples of computation of steady supersonic flow with discontinuities are presented here, namely:

(1) Collapse of a discontinuity with strong rarefaction waves and a very weak shock. The initial conditions are:

	To the left	To the right
u	4.732	4.218
v	0	1.048
$\bar{P}$	1	3.421
ρ	1	2.285

(2) The interaction of two shocks running into each other. The initial conditions are:

	To the left	The middle	To the right
u	4.218	4.732	4.218
v	1.048	0	− 1.048
$\bar{P}$	3.421	1	3.421
ρ	2.285	1	2.285

In Fig. 1 the pressure for example 1 is plotted as a function of y for several values of x. The creation and expansion of the rarefaction wave is seen on the right side of the figure. On the left, the scale along the P axis is made one hundred times greater, so that a very weak shock can be seen (at $x = 160$).

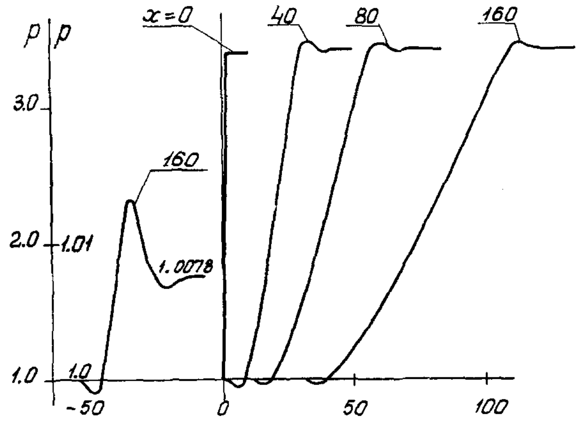

FIG. 1. The pressure distributions at lines of const $\bar{x}$ for the collapse of the discontinuity. The scale on the left is 100 times greater to make it possible to see very weak shocks.

In Fig. 2 and 3, respectively, the velocity components u and v in example 2 are plotted for several values of x during the interaction.

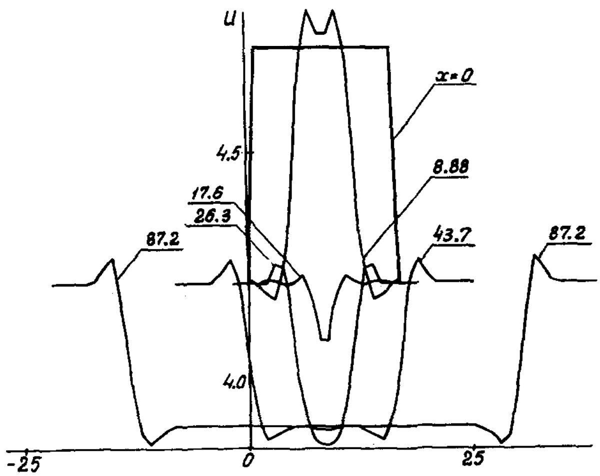

FIG. 2. The distributions of the velocity component u at lines of const $\bar{x}$ for two colliding shocks.

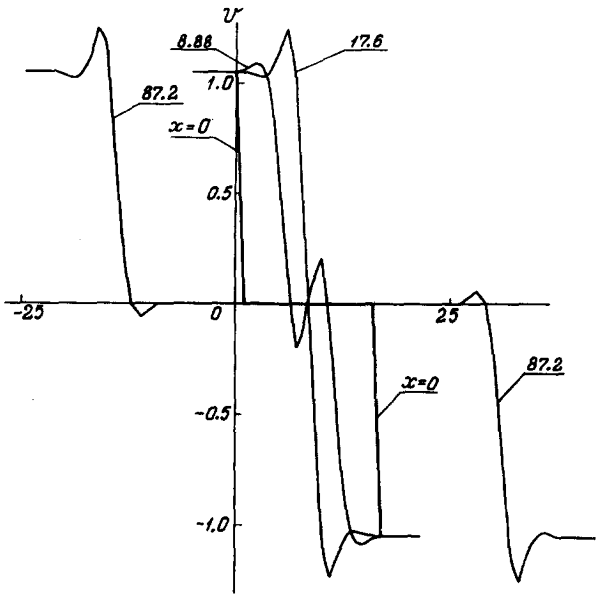

FIG. 3. The distribution of the velocity component v at lines of const $\bar{\bar{x}}$ for two colliding shocks.

IV. THE THIRD-ORDER SCHEME FOR THE TWO-DIMENSIONAL CASE

The construction of the third-order scheme for three variables, that is, for the system

$$\frac{\partial w}{\partial t} = \frac{\partial F(w, x, t)}{\partial x} + \frac{\partial G(w, x, t)}{\partial x} + f(w, x, t) \tag{13}$$

leads to such unwieldy calculations that the search for a full family of solutions is found to be a too complicated a problem, even if a computer is used. Thus we shall consider here only a simpler system, namely

$$\frac{\partial w}{\partial t} = \frac{\partial F(w)}{\partial x} + \frac{\partial G(w)}{\partial y}. \tag{14}$$

Let us introduce in (x, y, t)-space the rectangular mesh with steps $\Delta x = h_1$, $\Delta y = h_2$, $\Delta t = \tau$ and denote for any function $\varphi(w, x, y, t)$:

$$\varphi_{m,l}^n = \varphi(w_{m,l}^n, x_m, y_l, t^n), \qquad w_{m,l}^n = w(x_m, y_l, t^n).$$

The expansion of $w(x_m, y_l, t^n + \tau)$ in powers of τ has the form (3) where the derivatives w_t, w_{tt}, w_{ttt} are expressed as follows:

$$\begin{aligned} w_t &= H = D_x F + D_y G \\ w_{tt} &= K = D_x(F_w H) + D_y(G_w H) \\ w_{ttt} &= L = D_x\{F_w K + F_{ww} H^2\} + D_y\{G_w K + G_{ww} H^2\}. \end{aligned} \tag{15}$$

Let us consider the difference scheme for system (14) which is constructed by analogy with scheme (2) for system (1). We write it in the following general form:

$$\begin{aligned} w^{[1]}_{m+1/2,l+1/2} &= \mu_x \mu_y w^{[0]}_{m+1/2} + \beta_1\{q_1 \mu_y\, \delta_x F + q_2 \mu_x\, \delta_y G\}^{[0]}_{m+1/2,l+1/2} \\ w^{[2]}_{m,l} &= [I + \omega_{20}(\delta_x{}^2 + \delta_y{}^2)]\, w^{[0]}_{m,l} \\ &\quad + \beta_{21}\{q_1 \mu_x\, \delta_x F + q_2 \mu_y\, \delta_y G\}^{[0]}_{m,l} \\ &\quad + \beta_{22}\{q_1 \mu_y\, \delta_x F + q_2 \mu_x\, \delta_y G\}^{[1]}_{m,l} \\ w^{[3]}_{m,l} &= [I + \omega_{30}(\delta_x{}^2 + \delta_y{}^2) + \gamma_{30}(\delta_x{}^4 + \delta_y{}^4)]\, w^{[0]}_{m,l} \\ &\quad + \beta_{31}\{(I + \omega_{31}\delta_x{}^2 + \theta_{31}\delta_y{}^2)\, q_1 \mu_x\, \delta_x F \\ &\quad + (I + \theta_{31}\delta_x{}^2 + \omega_{31}\delta_y{}^2)\, q_2 \mu_y\, \delta_y G\}^{[0]}_{m,l} \\ &\quad + \beta_{32}\{(I + \omega_{32}\delta_x{}^2 + \theta_{32}\delta_y{}^2)\, q_1 \mu_y\, \delta_x F \\ &\quad + (I + \theta_{32}\delta_x{}^2 + \omega_{32}\delta_y{}^2)\, q_2 \mu_x\, \delta_y G\}^{[1]}_{m,l} \\ &\quad + \beta_{33}\{(I + \theta_{33}\delta_y{}^2)\, q_1 \mu_x\, \delta_x F + (I + \theta_{33}\delta_x{}^2)\, q_2 \mu_y\, \delta_y G\}^{[2]}_{m,l}\,. \end{aligned}$$

The meaning of the notations δ_x, δ_y, μ_x, μ_y is obvious.

In the same way as for Eq. (5) we obtain the expansion of $w^{[1]}_{m+1/2,l+1/2}$, $w^{[2]}_{m,l}$, $w^{[3]}_{m,l}$ in powers of τ, and after comparison of corresponding terms we obtain for the parameters the same values as in system (7). This result is evident for the first seven equations, but the coinciding of the equations for the parameter θ_{rs} is rather unexpected.

The coefficient γ_{30} does not influence the accuracy, but is essential for stability.

As in the one-dimensional case, we obtain a very simple scheme if we put $\omega_{20} = \beta_{21} = \theta_{31} = 0$. Then $\beta_1 = \frac{1}{3}$, $\beta_{22} = \frac{2}{3}$, $\beta_{31} = \frac{1}{4}$, $\beta_{33} = \frac{3}{4}$, $\omega_{31} = -\frac{2}{3}$, $\omega_{30} = \beta_{32} = \theta_{33} = 0$.

This scheme was also proposed by S. Burstein [4].

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